On the Calculation of the Generating Functions and Tables of Groundforms for Binary Quantics.

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THE object of this paper is to give an account of the methods, due to Professors Cayley and Sylvester, of calculating the generating functions pertaining to binary quantics and thence determining the number of fundamental invariants and covariants of any order and degree. As it will not very greatly increase the length of the paper, I shall endeavor, besides giving the processes of calculation, to present a connected view, though not a complete discussion, of the subject. It seems desirable to make some remarks at the outset on the terms employed, though these are, for the most part, well understood.

The *degree* of any function is its degree in the coefficients of the quantic, the *order* is its degree in the variables. The term *covariant* will be regarded, whenever convenient, as including invariants, the latter being covariants of the order zero. A *differentiant* is a symmetric function of the differences of the roots; since the source of any covariant (the coefficient of the highest power of x in the covariant) is a differentiant, and since the covariant is completely determined by its source, the discovery of covariants is reduced to that of differentiants. A *groundform* is an irreducible or fundamental invariant or covariant, i. e. one that is not a rational integral function of invariants and covariants of lower degrees and orders.

The symbol $(w:i,j)$ is employed to denote the number of ways in which the number w can be composed as the sum of j of the numbers $0, 1, 2, \ldots . i$ (repetitions being allowed); or, in other words, the number of ways in which w can be composed of j or fewer positive integers none greater than i. Putting $g = ij - 2w$, it will often be convenient to write $(i,j:g)$ instead of the above symbol, so that we have

$$(w:i,j) = (i,j:ij-2w); \qquad (i,j:g) = \left(\frac{ij-g}{2}:i,j\right).$$

SINGLE QUANTICS.

For simplicity, we shall first consider exclusively the case of a single quantic. The methods are all based upon the following fundamental theorem:

The number of linearly independent differentiants of the weight w and degree j, belonging to a quantic of the order i, is $(w:i,j)-(w-1:i,j)$. We shall use $\Delta(w:i,j)$ to denote $(w:i,j)-(w-1:i,j)$.

A differentiant of weight w and degree j, belonging to a quantic of order i, is the source of a covariant whose order is $ij-2w$, say g; so that the above theorem may be thus stated:

The number of linearly independent covariants of the order g and degree j, belonging to a quantic of the order i, is $(i,j:g)-(i,j:g+2)$. We shall use $\Delta(i,j:g)$ to denote $(i,j:g)-(i,j:g+2)$.

$(w:i,j)$ is the coefficient of c^jz^w in the development of

$$\frac{1}{(1-c)\,(1-cz)\,(1-cz^2)\ldots\ldots(1-cz^i)} \tag{1}$$

in ascending powers of c; or, putting $z=x^{-2}$, $c=ax^i$, we may say that $(w:i,j)$, $=(i,j:g)$, is the coefficient of a^jx^g in the development of

$$\frac{1}{(1-ax^i)\,(1-ax^{i-2})\ldots\ldots(1-ax^{-i+2})\,(1-ax^{-i})} \tag{2}$$

in ascending powers of a;

hence $\Delta(i,j:g)$ is the coefficient of a^jx^g in the development of

$$\frac{1-x^{-2}}{(1-ax^i)\,(1-ax^{i-2})\ldots\ldots(1-ax^{-i+2})\,(1-ax^{-i})} \tag{3}$$

in ascending powers of a.

From (1) is also deduced Euler's theorem that $(w:i,j)$ is the coefficient of z^w in the development of

$$\frac{(1-z^{j+1})\,(1-z^{j+2})\ldots\ldots(1-z^{j+i})}{(1-z)\,(1-z^2)\ldots\ldots(1-z^i)}, \tag{4}$$

so that $\Delta(w:i,j)$ is the coefficient of z^w in the development of

$$\frac{(1-z^{j+1})\,(1-z^{j+2})\ldots\ldots(1-z^{j+i})}{(1-z^2)\ldots\ldots(1-z^i)}. \tag{5}$$

The fraction (3) is a generating function in which the coefficient of a^jx^g is the number of linearly independent covariants of the degree j and order g, to a quantic of the order i; but it is far from being well adapted to calculation, and moreover furnishes no means of determining *the complete system of groundforms* of the quantic. Before describing the methods of constructing generating functions

which *do* serve this important purpose, it will be well to make some remarks upon the connection between the determination of the groundforms and that of the number of linearly independent covariants of a given degree and order.

Tamisage.

To begin with an example: Suppose we have found that the quintic has one irreducible invariant of each of the degrees $4, 8, 12$, and no other irreducible invariant of a degree lower than 16; and let it be proposed to find the number of irreducible invariants of the degrees 16 and 18. We can find by actually considering the partitions, or otherwise, that $\Delta(5, 16:0) = 4$; hence there are four linearly independent invariants of degree 16. Now, precisely four invariants of degree 16 can be formed from the lower irreducible invariants (which latter we may denote as $(4.0), (8.0)$ and (12.0)), namely, $(4.0)^4$, $(4.0)^2(8.0)$, $(4.0)(12.0)$ and $(8.0)^2$; hence, unless we suppose the previously found irreducible invariants to be connected by a syzygetic relation of degree 16, we conclude that there is no new — i. e. no irreducible — invariant of degree 16. Also $\Delta(5, 18:0) = 1$, so that there is one invariant of degree 18; and since no invariant of that degree can be formed from the lower invariants, we *know* that there is an irreducible invariant of degree 18. We should have made the same inference if $\Delta(5, 18:0)$ had not been 1, but had exceeded by 1 the number of ways in which invariants of degree 18 could be formed from lower irreducible invariants, provided we assumed, as before, that those lower invariants were not connected by any syzygetic relation of the degree in question. Again, keeping to the example of the quintic, suppose we have found that there is one irreducible covariant of each of the following types:

$$(4.0), (8.0), (5.1), (7.1), (2.2), (6.2), (8.2), (3.3), (5.3), (9.3), (4.4), (6.4)$$

where $(m.n)$ represents a covariant of degorder $(m.n)$, and that there are no other irreducible covariants whose degree is less than 10 and order not greater than 4; and let it be proposed to find the number of irreducible covariants of degree 10, order 4. We find $\Delta(5, 10:4) = 3$, so that there are three linearly independent covariants of degorder (10.4); and covariants of this degorder can be formed by compounding lower irreducibles in the following four ways:

$$(4.0)(6.4), (5.1)(5.3), (7.1)(3.3), (2.2)(8.2).$$

In this case, then, the number of compound forms exceeds by 1 the total number of linearly independent forms of the type in question, (10.4); hence there must be at least *one* syzygy of degorder (10.4) connecting the lower groundforms:

but we are not compelled to assume that *more* than one such syzygy exists; and assuming that the one *necessary* syzygy is the *only* one, we conclude that the three linearly independent covariants of degorder (10.4) are compounded ones, and that there is no groundform of that order.

Stated in general terms, the matter stands as follows: By means of the fundamental theorem,* we can ascertain the number (say α) of *linearly independent* covariants of degree j and order g; suppose now that we also know the number of *irreducible* covariants of every type whose degree is below j and whose order is not higher than g; and suppose there are β ways of producing a covariant of the given type (degorder $(j.g)$) by multiplying together these irreducible covariants. Then, if β is less than α, there are evidently *at least* $\alpha-\beta$ irreducible covariants of the given type; if β is equal to or greater than α, there *may* not be any irreducible covariants of the given type. In fact, if the β possible compound forms of the type are linearly independent, there remain $\alpha-\beta$ forms which are *not* compounds, i. e. which are groundforms; if β is greater than α, the β forms can not *all* be linearly independent; but α of them may be independent, and if they are, no uncompounded forms of the type remain. We assume (in the absence of demonstration, but with the support of very strong inductive evidence) that there never are groundforms and syzygies of one and the same degorder. Granting this fundamental postulate, the number of groundforms of the type considered is $\alpha-\beta$ if $\alpha>\beta$; if $\alpha \overline{\overline{<}} \beta$ the number is zero. Thus, if the numbers of linearly independent covariants of every degree up to j and of every order up to g be known, the numbers of the groundforms within the same limits are obtained by a process which Professor Sylvester calls *tamisage*, and of which it may be desirable to give an example.

Take the case of the octavic; the numbers of linearly independent forms of all degrees to the 7th and of all orders to the 8th are shown in the first of the following tables; the numbers of groundforms within the same limits are shown in the second table. The second table is deduced from the first as follows: We proceed regularly down the successive columns, writing in the corresponding places of the second table the number of groundforms of each type. Thus, (2.0) and (3.0) are groundforms, there being no lower forms out of which they can be compounded; one (4.0) can be produced by multiplying (2.0) by itself, so that the number of groundforms of this type is $2-1=1$; (2.0) (3.0) giving (5.0), we have again $2-1=1$; $(2.0)^3$, (2.0) (4.0), and $(3.0)^2$ each give one (6.0), so that the number of groundforms of this type is $4-3=1$; $(2.0)^2$ (3.0), (2.0) (5.0), (3.0) (4.0)

* Which, it is to be remembered, has been rigorously demonstrated by Professor Sylvester. See Borchardt's Journal, Vol. LXXXV. (1878), p. 104, and Philos. Mag., Vol. V. (1878), p. 178.

each giving (7.0), we have $4-3=1$ groundform of this type. We then proceed to the next column, deducting, as before, from each number in the first table the number of compound forms of the corresponding type that can be produced by the previously found groundforms; so that at any stage of the process we know

LINEARLY INDEPENDENT FORMS.

Degree.	Order. 0	2	4	6	8
1					1
2	1		1		1
3	1		1	1	2
4	2		3	1	4
5	2	1	4	3	6
6	4	1	7	5	11
7	4	3	10	9	16

GROUNDFORMS.

Degree.	Order. 0	2	4	6	8
1					1
2	1		1		1
3	1		1	1	1
4	1		2	1	1
5	1	1	2	2	1
6	1	1	2	3	1
7	1	2	2	3	0

the number of linearly-independent forms of a certain type and the number of groundforms of all lower types. Thus, suppose we have determined all the numbers in the second table except the last one, this is found as follows:

$$(2.0)\,,(2.0)^2,(2.0)^3,(2.0)\,(3.0)\,,(2.0)\,(4.0)\,,(3.0)\,,(3.0)^2,(4.0)\,,(5.0)\,,(6.0)$$

combined respectively with

$$(5.8)\,,(3.8)\,,(1.8)\,,\quad(2.8)\quad,\quad(1.8)\quad,\ (4.8),(1.8)\,,(3.8)\,,(2.8)\,,(1.8)$$

give 10 forms of the type (7.8), there being one groundform of each of the types above employed; $(3.0)t(2.4)^2$ and (2.0) (2.4) (3.4) likewise give 2; while the groundform (2.4) combined with each of the two groundforms (5.4), and the groundform (3.4) combined with each of the two groundforms (4.4) give in all 4 forms of the type (7.8). Thus, the total number of compound forms is $10+2+4=16$, and 16 being also the number of linearly independent forms, we see that there are no groundforms of the type in question.

This process of tamisage is evidently very long and tedious, the above example being of the simplest, and the labor plainly becoming greater at every step; moreover, however far we may carry the process we have no assurance that there are no groundforms beyond: the series representing the numbers of linearly independent forms being evidently infinite in extent. With the generating functions we are about to consider, the labor of tamisage is very greatly abridged, and, what is of essential importance, the field in which it has to be applied is finite.

Sylvester's First Method.

The above "*crude form*" (3) of the generating function is

$$\phi(x) = \frac{1 - x^{-2}}{(1 - ax^{i})(1 - ax^{i-2}) \ldots (1 - ax^{-i+2})(1 - ax^{-i})};$$

consider the decomposition of this into partial fractions, with reference to x. To any factor $1 - ax^{\lambda}$ of the denominator, λ being positive, will correspond λ partial fractions of the form

$$\frac{A}{1 - \rho a^{\frac{1}{\lambda}} x},$$

where A is a function of a, and ρ a λth root of unity; these λ fractions added together will give a fraction of the form

$$\frac{A_0 + A_1 x + A_2 x^2 + \ldots + A_{\lambda-1} x^{\lambda-1}}{1 - ax^{\lambda}},$$

expanding which in ascending powers of a, we get also ascending powers of x, beginning with the 0th power. But to any factor $1 - ax^{-\lambda}$ will correspond λ partial fractions of the form

$$\frac{A}{x - \rho a^{\frac{1}{\lambda}}};$$

the sum of these λ fractions is of the form

$$\frac{A_0 + A_1 x + A_2 x^2 + \ldots + A_{\lambda-1} x^{\lambda-1}}{x^{\lambda} - a},$$

i. e.

$$\frac{1}{x^{\lambda}} \cdot \frac{A_0 + A_1 x + A_2 x^2 + \ldots + A_{\lambda-1} x^{\lambda-1}}{1 - ax^{-\lambda}},$$

and this, when expanded in ascending powers of a, gives only negative powers of x, which are irrelevant to the question in hand. We can therefore obtain a generating function whose development shall coincide with that of $\phi(x)$ as far as the terms containing non-negative powers of x are concerned, by calculating the partial fractions corresponding to those factors of the denominator of $\phi(x)$ in which the exponent of x is positive, and taking the sum of these partial fractions.

The fraction corresponding to $1 - ax^{\lambda}$ is, as before stated,

$$\Sigma \frac{A}{1 - \rho a^{\frac{1}{\lambda}} x},$$

where ρ is to be successively every λth root of unity. Denoting by $\phi_{\lambda}(x)$ what $\phi(x)$ becomes when the factor $1 - ax^{\lambda}$ is struck out of the denominator, and writing α for $\rho a^{\frac{1}{\lambda}}$, it is seen at once that the value of A is $\frac{1}{\lambda}\phi_{\lambda}(\alpha^{-1})$; i. e.

$$\lambda A = \frac{1-\alpha^2}{(1-\alpha^{-i+\lambda})(1-\alpha^{-i+2+\lambda})\ldots\ldots(1-\alpha^{-2})(1-\alpha^2)\ldots\ldots(1-\alpha^{i+\lambda})}$$

$$= (-)^{\frac{i-\lambda}{2}} \frac{\alpha^{\frac{i-\lambda}{2}\cdot\frac{i-\lambda+2}{2}}(1-\alpha^2)}{(1-\alpha^{i-\lambda})(1-\alpha^{i-2-\lambda})\ldots\ldots(1-\alpha^2)(1-\alpha^2)\ldots\ldots(1-\alpha^{i+\lambda})}$$

$$= (-)^{\frac{i-\lambda}{2}} \frac{\alpha^{\frac{i-\lambda}{2}\cdot\frac{i-\lambda+2}{2}}(1-\alpha^2)}{(1-\alpha^2)^2(1-\alpha^4)^2(1-\alpha^6)^2\ldots\ldots(1-\alpha^{i-\lambda})^2(1-\alpha^{i-\lambda+2})(1-\alpha^{i-\lambda+4})\ldots\ldots(1-\alpha^{i+\lambda})}\,.$$

Any factor $1-\alpha^m$ of this denominator is a divisor of $1-\alpha^{\lambda k}$, where λk is the least common multiple of m and λ, giving a quotient $1+\alpha^m+\alpha^{2m}+\ldots\ldots$; multiplying numerator and denominator by the like quotients for all the factors, the denominator becomes a product of factors of the form $1-\alpha^k$, and the numerator alone involves α. Thus we have

$$\frac{A}{1-\alpha x} = \frac{1}{\lambda}\cdot\frac{c_0+c_1\alpha+c_2\alpha^2+\ldots\ldots}{(1-\alpha x)(1-\alpha^k)(1-\alpha^{k'})\ldots\ldots}$$

and hence

$$\Sigma\frac{A}{1-\alpha x} = \frac{1}{\lambda}\cdot\frac{\Sigma\,(c_0+c_1\alpha+c_2\alpha^2+\ldots\ldots)(1+\alpha x+\alpha^2x^2+\ldots\ldots+\alpha^{\lambda-1}x^{\lambda-1})}{(1-\alpha x^{\lambda})(1-\alpha^k)(1-\alpha^{k'})\ldots\ldots}$$

It only remains to collect those terms of the numerator in which the exponent of α is a multiple of λ (the others disappearing in the summation), and we have $\Sigma\dfrac{A}{1-\alpha x}$ expressed in the form

$$\frac{L_0+L_1x+\ldots\ldots+L_{\lambda-1}x^{\lambda-1}}{(1-\alpha x^{\lambda})(1-\alpha^k)(1-\alpha^{k'})\ldots\ldots}\,.$$

The sum of the several fractions obtained in this way is evidently of the form

$$\frac{C_0+C_1x+C_2x^2+\ldots\ldots}{(1-\alpha x^i)(1-\alpha x^{i-2})\ldots\ldots(1-\alpha^k)(1-\alpha^{k'})\ldots\ldots}\,,$$

where, it may be remarked in passing, the numerator is of lower degree in x than the denominator. This fraction, written in its lowest terms,* is called the *reduced form* of the generating function.

Representative Form of the Generating Function.

In general the reduced form does not serve directly for obtaining the numbers of the groundforms. It would serve this purpose — as we shall presently show — if the factors of the denominator all corresponded to groundforms, i. e. if to each factor $1-\alpha^r x^s$ (where s may be 0) in the denominator there corre-

* Not always *strictly* in its lowest terms, but in the lowest form having only factors of the types $1-\alpha^r$, $1-\alpha^r x^s$ in the denominator.

sponded a groundform of degorder $(r \,.\, s)$; and in this case the factors of the denominator are said to be representative. If this is not the case, we multiply numerator and denominator by such factors as will make the denominator a product of representative factors exclusively; such factors exist for all the quantics whose generating functions have been calculated, with the exception of the septimic, in which there is one factor in the denominator which cannot be converted into a representative one. Nor is it difficult to find these factors. Those containing x are disposed of at once; for $1 - ax^i$ represents the quantic itself, and there being no other covariant of the first degree, the factors $1 - ax^{i-2}$, $1 - ax^{i-4}$, etc. are not representative; but since every quantic has a series of covariants (obviously irreducible) of the second degree and of orders $2(i-2)$, $2(i-4)$, etc. the factors $1 - ax^{i-2}$, $1 - ax^{i-4}$, etc. will be converted into representative ones by multiplication by $1 + ax^{i-2}$, $1 + ax^{i-4}$, etc. As to any factor $1 - a^r$ independent of x, it is easy to infer from the generating function itself whether it is representative or not, i. e. whether there is or is not an irreducible *in*variant of the degree r;* and if not, we have to find whether there is one of a degree that is a multiple of r, say mr; in which case we multiply numerator and denominator by $\frac{1 - a^{mr}}{1 - a^r}$. For all the quantics that have been considered (with the single exception above mentioned), it is found that the non-representative factors become representative on merely *doubling* the exponent.—When these multiplications have been performed, so that every factor in the denominator is representative, the generating function is said to be in the *representative form*.†

Mode of obtaining the Table of Groundforms.

We shall now show that when the generating function is in a representative form, the groundforms consist of those represented by the factors of the denominator, together with those obtained by tamisage (with a certain modification) upon the numerator.

Let $L = \Sigma m_{j,g} a^j x^g$, where $m_{j,g}$ is the number of linearly independent covariants of degorder $(j \,.\, g)$. Then if there be a covariant, say V, of degorder $(r \,.\, s)$, the number of linearly independent covariants of degorder $(j \,.\, g)$ in which

* If a factor is repeated, it must represent a distinct groundform each time; e. g. if the denominator contains $(1 - a^6)^2$ it is not representative unless there are at least *two* irreducible invariants of the degree 6.

† Or, rather, *a* representative form. Others may be obtained by multiplying numerator and denominator by representative factors.

V enters as a factor is evidently exactly equal to the whole number of linearly independent covariants of degorder $(j-r\,.\,g-s)$, i. e. equal to the coefficient of $a^j x^g$ in $a^r x^s L$. Hence the excess of the whole number of linearly independent covariants of degorder $(j\,.\,g)$ over the number of such covariants in which V enters as a factor is the coefficient of $a^j x^g$ in $(1-a^r x^s)\,L$. It follows hence that if V is a groundform, the *other* groundforms can be got by tamisage from $(1-a^r x^s)\,L$, just as *all* the groundforms would be got from L.

In extending this result, so that we may be enabled to multiply L by more than one factor of the form $1-a^r x^s$, it is necessary to appeal to the fundamental postulate, that if there are syzygies of any given degorder, there are no groundforms of that degorder (p. 131, l. 17). Consider the linearly independent covariants of degorder $(j\,.\,g)$; suppose that the groundforms of the quantic are not connected by any syzygies of this degorder; and imagine these covariants distributed, with reference to any k groundforms $V_1, V_2, V_3 \ldots\ldots V_k$, of degorders $(r_1\,.\,s_1), (r_2\,.\,s_2), (r_3\,.\,s_3) \ldots\ldots (r_k\,.\,s_k)$ into classes, as follows:—

m_k covariants containing the product of all the $k\,V$'s as a factor.*
m_{k-1} " " any $(k-1)$ and only $(k-1)$ V's.

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m_2 covariants containing two and only two V's.
m_1 " " one " " one V.
m_0 " " no V.

Then it is easily seen that the coefficient of $a^j x^g$

in L	is	$m_0 + m_1 + m_2 + m_3 + \ldots\ldots + m_k$
in $-\Sigma a^r x^s L$	is	$-\frac{1}{1}m_1 - \frac{2}{1}m_2 - \frac{3}{1}m_3 - \ldots\ldots - \frac{k}{1}m_k$
in $\Sigma a^{r+r'} x^{s+s'} L$	is	$\frac{2.1}{1.2}m_2 + \frac{3.2}{1.2}m_3 + \ldots\ldots + \frac{k(k-1)}{1.2}m_k$
in $-\Sigma a^{r+r'+r''} x^{s+s'+s''} L$	is	$-\frac{3.2.1}{1.2.3}m_3 - \ldots\ldots - \frac{k(k-1)(k-2)}{1.2.3}m_k$
	etc.	etc.

For the coefficient in L is the total number of linearly independent covariants of degorder $(j\,.\,g)$; the coefficient in $\Sigma a^r x^s L$ is the number of linearly independent covariants which multiplied by any V give rise to a covariant of

* Whether any V appears to the first or to a higher power is indifferent to this argument throughout.

degorder $(j.g)$, so that in this coefficient every such compounded covariant is counted once for each distinct V that it contains; the coefficient in $\Sigma a^{r+r'}x^{s+s'}L$ is the number of linearly independent covariants which multiplied by any two distinct V's give rise to a covariant of degorder $(j.g)$, so that in this coefficient every such compounded covariant is counted once for each distinct binary combination of the V's that it contains; and so on.

By addition of these results, we see that m_0 is the coefficient of $a^j x^g$ in

$$\begin{aligned} &L(1 - \Sigma a^r x^s + \Sigma a^{r+r'}x^{s+s'} - \Sigma a^{r+r'+r''}x^{s+s'+s''} + \ldots .) \\ &= L(1 - a^{r_1}x^{s_1})(1 - a^{r_2}x^{s_2})(1 - a^{r_3}x^{s_3}) \ldots . (1 - a^{r_k}x^{s_k}); \end{aligned}$$

i. e. *if the groundforms of the quantic are not connected by any syzygies of the degorder* $(j.g)$, *the coefficient of* $a^j x^g$ *in* $L(1 - a^{r_1}x^{s_1})(1 - a^{r_2}x^{s_2}) \ldots . (1 - a^{r_k}x^{s_k})$ *is the number of linearly independent covariants of degorder* $(j.g)$ *containing none of the groundforms* $V_1, V_2, \ldots . V_k$.

If there *are* syzygies of the degorder $(j.g)$, we have to distinguish between two cases. (For brevity, we shall denote $L(1 - a^{r_1}x^{s_1})(1 - a^{r_2}x^{s_2}) \ldots . (1 - a^{r_k}x^{s_k})$ by Λ.)

1° Let us suppose that no syzygies existed among covariants of lower degorders which are raised to the degorder $(j.g)$ by multiplication by groundforms. Then the coefficient of $a^j x^g$ in L has been diminished by what would be the number of covariants obtained by compounding with the V's if these compounds were linearly independent; so that if they are *not* all linearly independent the diminution has been too great: in other words, the coefficient of $a^j x^g$ is either equal to or less than the number of linearly independent covariants of degorder $(j.g)$ containing none of the V's. Now if we were to perform the operation of tamisage upon L, the existence of syzygies of the degorder $(j.g)$ would (according to the fundamental postulate) be indicated by the coefficient of $a^j x^g$ being rendered negative by the operation. Likewise, if the coefficient of $a^j x^g$ were exactly equal to the number of linearly independent covariants of degorder $(j.g)$ not containing the V's, it would be rendered negative by tamisage upon the coefficients in Λ; and *à fortiori* the *actual* coefficient (which has just been proved to be less than if not equal to the above number) will be rendered negative. Hence in this first case tamisage upon Λ gives no groundforms of degorder $(j.g)$, which is correct; and, moreover, indicates the existence of syzygies of that degorder.

2° Suppose that syzygies of the degorder $(j.g)$ exist which are not groundsyzygies, but are obtained from lower syzygies through multiplication by groundforms or by combinations of groundforms. Then, from what has been

said under 1°, the process of tamisage must have revealed these lower syzygies at their inception. We have, therefore, only to see whether $(j \,.\, g)$ is a degorder which can be formed by composition of a degorder in which a syzygy has been found to occur, with the degorder of any groundform or combination of groundforms (*not excepting the* V'*s*); if it is, we infer, by the fundamental postulate, that no groundform of the degorder $(j \,.\, g)$ exists.

Let us now return to the representative generating function. This is of the form

$$\frac{N}{(1-a^{r_1}x^{s_1})\,(1-a^{r_2}x^{s_2})\,\ldots\ldots(1-a^{r_k}x^{s_k})}$$

(some of the s's are 0) where the factors of the denominator correspond to groundforms. The development of this fraction is identical with what we have above called L, so that N is identical with $L\,(1-a^{r_1}x^{s_1})\,(1-a^{r_2}x^{s_2})\ldots\ldots(1-a^{r_k}x^{s_k})$. The foregoing discussion shows that the groundforms additional to those represented in the denominator are to be found as follows:—

Operate by tamisage upon the coefficients of N.

a) If the coefficient of $a^j x^g$, as reduced by tamisage, is a positive number or 0 (and provided the coefficient is not rejected in virtue of b)), this reduced coefficient is the number of groundforms of degorder $(j \,.\, g)$ * and there are no syzygies of that degorder.

b) If the coefficient of $a^j x^g$, as reduced by tamisage, is negative, there are syzygies and no groundforms of the degorder $(j \,.\, g)$; moreover, there are syzygies and no groundforms in all higher degorders derivable from $(j \,.\, g)$ by compounding $(j \,.\, g)$ with the degorders of groundforms (the groundforms represented in the denominator, as well as those found from the numerator), so that the coefficients corresponding to such higher degorders are to be rejected.

The numerator being finite, the above operation of modified tamisage is finite, which is a matter of essential importance. The practical advantage in having to operate upon the numbers appearing in N, which are of necessity very much smaller than those in the infinite series, is obvious.

Constitution of the Generating Function as first obtained.

We shall now go back and look more closely at the form of the generating function as primarily given by Sylvester's first method; for an examination of

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* Additional, of course, to any of the same degorder that may be represented in the denominator.

this form gives rise to two other methods differing materially from the former and from each other.

It will be best to mention, in the first place, a distinction between quantics of even and quantics of odd orders which has hitherto been passed over for the sake of uniformity. For even quantics the crude form, $\phi(x)$, involves only even powers of x, and may be treated as a function of x^2, which of course greatly abridges the work. In examining the form of the generating function, we shall treat separately quantics of even and quantics of odd orders.

Let us take, first, a quantic of even order, $2n$, so that

$$\phi(x) = \frac{1 - x^{-2}}{(1 - ax^{2n})(1 - ax^{2n-2}) \ldots . (1 - a) \ldots . (1 - ax^{-2n+2})(1 - ax^{-2n})}$$
$$= \frac{1 - u^{-1}}{(1 - au^{n})(1 - au^{n-1}) \ldots . (1 - a) \ldots . (1 - au^{-n+1})(1 - au^{-n})} .$$

As before, we have to find the sum of the partial fractions corresponding to those factors of the denominator in which the exponent of u is positive. The fraction corresponding to $1 - au^{\lambda}$ is

$$\Sigma \frac{A}{1 - \rho a^{\frac{1}{\lambda}} u},$$

and we find, in the same manner as before, denoting $\rho a^{\frac{1}{\lambda}}$ by α, that

$$\lambda A = (-)^{n-\lambda} \frac{\alpha^{\frac{(n-\lambda)(n-\lambda+1)}{2}} (1 - \alpha)}{(1 - \alpha)^2 (1 - \alpha^2)^2 (1 - \alpha^3)^2 \ldots . (1 - \alpha^{n-\lambda})^2 (1 - \alpha^{n-\lambda+1}) \ldots . (1 - \alpha^{n+\lambda})} .$$

Here λ is some number between 1 and n (both inclusive). When $\lambda = n$, the above denominator (after cancelling the factor $1 - \alpha$) is

$$(1 - \alpha^2)(1 - \alpha^3) \ldots . (1 - \alpha^{2n-1})(1 - \alpha^2),$$

which is evidently contained in

$$(1 - a^2)^2 (1 - a^3) \ldots . (1 - a^{2n-1});$$

when λ is less than n, the denominator (after cancellation of $1 - \alpha$) is

$$(1 - \alpha^2)(1 - \alpha^3) \ldots . (1 - \alpha^{n+\lambda})(1 - \alpha)(1 - \alpha^2) \ldots . (1 - \alpha^{n-\lambda}),$$

and this is also contained in

$$(1 - a^2)^2 (1 - a^3) \ldots . (1 - a^{2n-1}).$$

For $(1 - \alpha^2)(1 - \alpha^3) \ldots . (1 - \alpha^{n+\lambda})$

is contained in $\quad (1 - a^2)(1 - a^3) \ldots . (1 - a^{n+\lambda});$

and $(1 - \alpha)(1 - \alpha^2) \ldots . (1 - \alpha^{n-\lambda}),$

being contained in $(1-a^{n+\lambda})(1-a^{n+\lambda+1})\ldots\ldots(1-a^{2n-1})$

and also in $(1-a^{n-\lambda})(1-a^{n+\lambda+1})\ldots\ldots(1-a^{2n-1})$,

is contained in their difference, and therefore in

$$(1-a^{2\lambda})(1-a^{n+\lambda+1})\ldots\ldots(1-a^{2n-1}),$$

which is itself contained in $(1-a^2)(1-a^{n+\lambda+1})\ldots\ldots(1-a^{2n-1})$.

Thus we see that the partial fractions will have for a common denominator

$$(1-a^2)^2(1-a^3)(1-a^4)\ldots\ldots(1-a^{2n-1})(1-ax^2)(1-ax^4)\ldots\ldots(1-ax^{2n}).$$

It remains to notice the degree of the numerator in a and x. The degree in x is necessarily less by at least 2 (since only even powers appear) than the degree of the denominator, and therefore cannot exceed $n(n+1)-2$. We shall show presently (see p. 141) that this is the *exact* degree. The degree in a must fall short of the degree of the denominator by $2n+1$ exactly; for in the value of $\frac{A}{1-au}$ we notice that the degree in a of the numerator falls short of that of the denominator by $\frac{(n+\lambda)(n+\lambda+1)}{2}$, which difference remains unaltered, of course, in the subsequent operations; now this difference of degree in a is equivalent to a difference of $\frac{(n+\lambda)(n+\lambda+1)}{2\lambda}$ in a, the least value of which, as λ takes the values $1, 2, \ldots\ldots n$, is attained when $\lambda = n$, and is $2n+1$: and since this minimum difference is reached *only* when $\lambda = n$, the term involving the corresponding power of a cannot be destroyed. Hence the degree in a of the numerator is $2+\big(2+3+\ldots\ldots+(2n-1)\big)+n-(2n+1)=2n(n-1)$.

Secondly, take a quantic of odd order, $2n+1$,* so that

$$\phi(x)=\frac{1-x^{-2}}{(1-ax^{2n+1})(1-ax^{2n-1})\ldots\ldots(1-ax)(1-ax^{-1})\ldots\ldots(1-ax^{-2n+1})(1-ax^{-2n-1})}$$

and

$$\lambda A=(-)^{\frac{2n+1-\lambda}{2}}\frac{a^{\frac{2n+1-\lambda}{2}\cdot\frac{2n+3-\lambda}{2}}(1-a^2)}{(1-a^2)^2(1-a^4)^2\ldots\ldots(1-a^{2n+1-\lambda})^2(1-a^{2n+3-\lambda})\ldots\ldots(1-a^{2n+1+\lambda})},$$

where λ is some odd number between 1 and $2n+1$, both inclusive. When $\lambda = 2n+1$, the above denominator (after cancellation of $1-a^2$) is

$$(1-a^4)(1-a^6)\ldots\ldots(1-a^{4n})(1-a^2),$$

which is contained in $(1-a^2)(1-a^4)(1-a^6)\ldots\ldots(1-a^{4n})$;

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* n not zero. What is here and in subsequent parts of the paper given concerning the form of the numerator does not in general apply to the linear quantic.

when λ is less than $2n+1$, the denominator (after cancellation of $1-a^2$) is

$$(1-a^4)(1-a^6)\ldots\ldots(1-a^{2n+1+\lambda})(1-a^2)(1-a^4)\ldots\ldots(1-a^{2n+1-\lambda})$$

which (as can be seen in the same way as the corresponding case for even quantics) is also contained in

$$(1-a^2)(1-a^4)(1-a^6)\ldots\ldots(1-a^{4n}).$$

Hence the partial fractions will have for a common denominator

$$(1-a^2)(1-a^4)(1-a^6)\ldots\ldots(1-a^{4n})(1-ax)(1-ax^3)\ldots\ldots(1-ax^{2n+1}).$$

The degree in x of the numerator is necessarily less by at least 1 than that of the denominator, and the degree in a is seen (in the same manner as for even quantics) to be less by exactly $2n+2$ than that of the denominator. We shall immediately show that the degree in x falls short of the degree of the denominator by exactly 2; hence the degree in x is $(n+1)^2-2$, and the degree in a is $4n^2+n-1$.

The proof that the degree in x of the numerator of the generating function is less by *exactly* 2 than that of the denominator applies to even and odd quantics alike, and is as follows. Denote the generating function obtained by

$$\frac{f(x)}{(1-ax^i)(1-ax^{i-2})\ldots\ldots};$$

this was obtained by adding together those partial fractions which corresponded to the factors in the denominator of $\phi(x)$ in which the exponent of x was positive: hence it is easy to see that

$$\phi(x)=\frac{f(x)}{(1-ax^i)(1-ax^{i-2})\ldots\ldots}-x^{-2}\frac{f(x^{-1})}{(1-ax^{-i})(1-ax^{-(i-1)})\ldots\ldots},$$

i. e. $1-x^{-2}=f(x)(1-ax^{-i})(1-ax^{-(i-2)})\ldots\ldots-x^{-2}f(x^{-1})(1-ax^i)(1-ax^{i-2})\ldots\ldots$

Now the degree in x of

$$x^{-2}f(x^{-1})(1-ax^i)(1-ax^{i-2})\ldots\ldots$$

is less by exactly 2* than that of

$$(1-ax^i)(1-ax^{i-2})\ldots\ldots;$$

hence, in virtue of the above identity, the degree in x of

$$f(x)(1-ax^{-i})(1-ax^{-(i-2)})\ldots\ldots,$$

* Since it is evident from the mode of formation that $f(x)$, and consequently $f(x^{-1})$, contains a term independent of x.

i. e. of $f(x)$, must also be less by exactly 2 than that of

$$(1-ax^i)(1-ax^{i-2})\ldots\ldots \quad \text{Q. E. D.}$$

The above results may be summed up as follows: That portion of the development of $\phi(x)$ which does not contain negative powers of x can be represented by a fraction, whose denominator for a quantic of order $i = 2n$ is

$$(1-a^2)^2(1-a^3)(1-a^4)\ldots\ldots(1-a^{2n-1})(1-ax^2)(1-ax^4)\ldots\ldots(1-ax^{2n})$$

and for a quantic of order $i = 2n+1$ is

$$(1-a^2)(1-a^4)(1-a^6)\ldots\ldots(1-a^{4n})(1-ax)(1-ax^3)\ldots\ldots(1-ax^{2n+1}).$$

In both cases, the exponent of the highest power of x that appears in the numerator is less by 2 than the degree in x of the denominator, and the exponent of the highest power of a that appears in the numerator is less by $i+1$ than the degree in a of the denominator.

It is to be observed that we have not proved that the generating function with the denominator above named is in its lowest terms, or even in the lowest terms consistent with the denominator being a product of factors of the forms $1-a^r$, $1-a^r x^s$; nor is it of any special importance whether the fraction is in its lowest terms or not. The difference between the degrees (in a and in x) of the numerator and denominator is of course unaltered by the introduction or suppression of common factors. As a matter of fact, it has been found that in the quantics of the orders 3, 5, 7, 9, 4, 8, a factor $1-a^2$ in the denominator as above given can be suppressed, but that in the quantics of the orders 2, 6, 10, no factor explicitly appearing in the denominator as above given is a divisor of the numerator.

Cayley's Method.

Instead of decomposing the crude generating function

$$\phi(x) = \frac{1-x^{-2}}{(1-ax^i)(1-ax^{i-2})\ldots\ldots(1-ax^{-i+2})(1-ax^{-i})}$$

into partial fractions with respect to x, this method proceeds by decomposing into partial fractions with respect to a. There will be simply $i+1$ such partial fractions, corresponding to the $i+1$ factors of the above denominator, these being linear in a. The fraction corresponding to $1-ax^\lambda$ (where λ is any one of the numbers $i, i-2, \ldots\ldots, -i+2, -i$) is

$$\frac{1-x^{-2}}{(1-x^{i-\lambda})(1-x^{i-\lambda-2})\ldots.(1-x^2)(1-x^{-2})\ldots.(1-x^{-i-\lambda})}\cdot\frac{1}{1-ax^\lambda}$$

$$=(-1)^{\frac{i+\lambda-2}{2}}\frac{x^{\frac{i+\lambda-2}{2}\cdot\frac{i+\lambda+4}{2}}(1-x^2)}{(1-x^2)(1-x^4)\ldots.(1-x^{i-\lambda})(1-x^2)(1-x^4)\ldots.(1-x^{i+\lambda})}\cdot\frac{1}{1-ax^\lambda}.$$

Now this decomposition does not enable us to separate the crude fraction into two parts, such that the expansion of one shall involve *only* negative powers of x and that of the other *no* negative powers of x; for it is plain that the expansion (in ascending powers of a) of every one of the partial fractions of the above form contains an infinite series of positive powers of x. But we now know that if in these expansions we reject the negative powers of x, the sum of the remaining portions is equal to a certain fraction whose denominator is known, and the degree in x and in a of whose numerator is also known. This required numerator is therefore identical with the product of the known denominator by the sum of those portions of the developments of the partial fractions which contain non-negative powers of x; hence we can obtain the numerator by developing each of the partial fractions in ascending powers of a and x, adding the results and multiplying the sum by the known denominator. And since this multiplier contains no negative powers of a or of x, we need go no farther in the expansions of the partial fractions than the known limits of the degrees in a and x of the numerator.

Thus, although the decomposition with respect to a introduces infinite series, we can, in virtue of our knowledge of the form of the result, arrive, by the use of finite portions of these series, at the same generating function as that obtained by Sylvester's method; and the work is in general considerably shorter by the present method. We shall now examine more closely the limits within which the work is confined.

Take, first, a quantic of order $i=2n$. We have seen that the generating function will have for its denominator

$$(1-a^2)^2(1-a^3)(1-a^4)\ldots.(1-a^{2n-1})(1-ax^2)(1-ax^4)\ldots.(1-ax^{2n}),$$

and that its numerator will be of the degree $q=n(n+1)-2$ in x and of the degree $p=2n(n-1)$ in a. Let μ be any one of the positive integers $1, 2, \ldots. n$. The partial fractions corresponding to $1-ax^{2\mu}$, $1-a$, $1-ax^{-2\mu}$, respectively, are

$$(-1)^{n+\mu-1}\frac{x^{(n+\mu-1)(n+\mu+2)}(1-x^2)}{(1-x^2)(1-x^4)\ldots.(1-x^{2n-2\mu})(1-x^2)(1-x^4)\ldots.(1-x^{2n+2\mu})}\cdot\frac{1}{1-ax^{2\mu}},$$

$$(-1)^{n-1}\frac{x^{(n-1)(n+2)}(1-x^2)}{(1-x^2)(1-x^4)\ldots.(1-x^{2n})(1-x^2)(1-x^4)\ldots.(1-x^{2n})}\cdot\frac{1}{1-a},$$

$$(-1)^{n-\mu-1}\frac{x^{(n-\mu-1)(n-\mu+2)}\,(1-x^2)}{(1-x^2)(1-x^4)\ldots.(1-x^{2n+2\mu})(1-x^2)(1-x^4)\ldots.(1-x^{2n-2\mu})}\cdot\frac{1}{1-ax^{-2\mu}}.$$

In the expansion of the first of the above expressions, the lowest exponent of x is $(n+\mu-1)(n+\mu+2)$, which is greater than q; hence we reject all the partial fractions corresponding to those factors of the denominator of the crude fraction in which the exponent of x is positive. The expansion of the first factor of the second expression is

$$(-1)^{n-1}\,x^{(n-1)(n+2)} + \text{terms of higher degree in } x;$$

accordingly, since $(n-1)(n+2)=q$, the portion of the expansion of the partial fraction corresponding to $1-a$ which we have to take into account is

$$(-1)^{n-1}\,x^q\,(1+a+a^2+\ldots.+a^p).$$

With regard to the third expression (the partial fraction corresponding to $1-ax^{-2\mu}$), we observe that the needed portion of the expansion of its second factor is

$$1+ax^{-2\mu}+a^2x^{-4\mu}+\ldots.+a^px^{-2p\mu},$$

so that we must expand the first factor as far as the power $x^{q+2p\mu}$, and multiply the result by the above series, rejecting in the product negative powers of x and also positive powers higher than q. Finally, we multiply the sum of the retained portions of all these expansions by the denominator

$$(1-a^2)^2\,(1-a^3)\,(1-a^4)\ldots.(1-a^{2n-1})\,(1-ax^2)\,(1-ax^4)\ldots.(1-ax^{2n});$$

retaining in the product only powers of a not higher than p and powers of x not higher than q, we have the required numerator.

For a quantic of odd order, $i=2n+1$, we find in like manner that we have to take account only of the partial fractions corresponding to those factors of the denominator of the crude fraction in which the exponent of x is negative. The denominator of the required generating function is

$$(1-a^2)\,(1-a^4)\,(1-a^6)\ldots.(1-a^{4n})\,(1-ax)\,(1-ax^3)\ldots.(1-ax^{2n+1});$$

the degrees of the numerator in a and x are $p=4n^2+n-1$, $q=(n+1)^2-2$. The partial fraction corresponding to $1-ax^{-2\mu-1}$ is

$$(-1)^{n-\mu-1}\frac{x^{(n-\mu-1)(n-\mu+2)}\,(1-x^2)}{(1-x^2)(1-x^4)\ldots.(1-x^{2n+2\mu+2})(1-x^2)(1-x^4)\ldots.(1-x^{2n-2\mu})}\cdot\frac{1}{1-ax^{-2\mu-1}};$$

we expand the first factor as far as the power $x^{q+(2\mu+1)p}$, and multiply the result by

$$1+ax^{-2\mu-1}+a^2x^{-4\mu-2}+\ldots.+a^px^{-2p\mu-p}.$$

Collecting the terms thus obtained for $\mu = 0, 1, 2, \ldots\ldots, n$, we multiply the sum by the denominator above given; the terms of the product within the assigned limits of degree constitute the required numerator.

In practice, we calculate the numerator only as far as the power $\frac{p}{2}$ or $\frac{p+1}{2}$ (say p') of a; a known symmetry enabling us to write the remaining half without calculation. The extent to which each of the expansions has to be carried is then given by putting p' for p in the preceding formulæ; and it is obvious that the abridgment of the work is very much more than half, even without taking account of the fact that the numerical coefficients in the expansions rapidly increase in magnitude. — The same symmetry would enable us to dispense with the calculation of half the powers of x instead of a, but it is plain that this would result in much less saving.

The symmetry referred to is proved as follows: —

Let the generating function for a quantic of order $2n+1$ be

$$F(x) = \frac{A_0 + A_1 x + \ldots\ldots + A_q x^q}{(1-ax)(1-ax^3)\ldots\ldots(1-ax^{2n+1})(1-a^\alpha)(1-a^\beta)\ldots\ldots},$$

and as before let p denote the highest power of a in the numerator. Then we have

$$\phi(x) = F(x) - x^{-2}F(-x)$$

i. e.

$$\begin{aligned}(1-x^{-2})(1-a^\alpha)(1-a^\beta)\ldots\ldots = {} & (A_0 + A_1x + \ldots\ldots + A_qx^q)(1-ax^{-1})(1-ax^{-3})\ldots\ldots(1-ax^{-2n-1}) \\ & - x^{-2}(A_0 + A_1x^{-1} + \ldots\ldots + A_qx^{-q})(1-ax)(1-ax^3)\ldots\ldots(1-ax^{2n+1}).\end{aligned}$$

It is to be noticed (see p. 142, l. 9) that $\alpha + \beta + \ldots\ldots = p + n + 1$, and that $1 + 3 + \ldots\ldots + (2n+1) = q + 2$; let us denote the number of the factors $1 - a^\alpha, 1 - a^\beta, \ldots\ldots$ by k. In the above identity, replace a by a^{-1} and x by x^{-1}, and multiply both sides by $(-1)^{n+1}a^{p+n+1}x^{-2}$; then, denoting by A'_μ what A_μ becomes when in it a is changed into a^{-1} and the result multiplied by a^p, the identity becomes

$$\begin{aligned}(-1)^{n-k}(1-x^{-2})(1-a^\alpha)(1-a^\beta)\ldots\ldots = {} & (A'_0x^q + A'_1x^{q-1} + \ldots\ldots + A'_q)(1-ax^{-1})(1-ax^{-3})\ldots\ldots(1-ax^{-2n-1}) \\ & - x^{-2}(A'_0x^{-q} + A'_1x^{-q+1} + \ldots\ldots + A'_q)(1-ax)(1-ax^3)\ldots\ldots(1-ax^{2n+1}).\end{aligned}$$

Comparing this with the former identity, we see that

$$A'_\mu = (-1)^{n-k}A_{q-\mu},$$

or, in other words, the coefficients of $a^\lambda x^\mu$ and $a^{p-\lambda}x^{q-\mu}$ in the numerator of $F(x)$ are equal in absolute value, and have like or opposite signs throughout, according as $n-k$ is even or odd.

In like manner, we have for a quantic of order $2n$

$$F(x) = \frac{A_0 + A_2x^2 + \ldots . + A_qx^q}{(1-ax^2)(1-ax^4)\ldots .(1-ax^{2n})(1-a^\alpha)(1-a^\beta)\ldots .},$$

whence

$$(1-x^{-2})(1-a^\alpha)(1-a^\beta)\ldots . = (A_0+A_2x^2+\ldots .+A_qx^q)(1-a)(1-ax^{-2})(1-ax^{-4})\ldots .(1-ax^{-2n})$$
$$-x^{-2}(A_0+A_2x^{-2}+\ldots .+A_qx^{-q})(1-a)(1-ax^2)(1-ax^4)\ldots .(1-ax^{2n}).$$

Denoting again by A'_μ what A_μ becomes when a is changed into a^{-1} and the result multiplied by a^p, we derive as before

$$(-1)^{n-k}(1-x^{-2})(1-a^\alpha)(1-a^\beta)\ldots . = (A'_0x^q+A'_2x^{q-2}+\ldots .+A'_q)(1-a)(1-ax^{-2})(1-ax^{-4})\ldots .(1-ax^{-2n})$$
$$-x^{-2}(A'_0x^{-q}+A'_2x^{-q+2}+\ldots .+A'_q)(1-a)(1-ax^2)(1-ax^4)\ldots .(1-ax^{2n}).$$

By comparing these identities we find

$$A'_\mu = (-1)^{n-k}A_{q-\mu};$$

i. e. the coefficients of $a^\lambda x^\mu$ and $a^{p-\lambda}x^{q-\mu}$ in the numerator of $F(x)$ are equal in absolute value, and of like or opposite signs according as $n-k$ is even or odd.

Professor Cayley has given a full statement of the calculation in the case of the seventhic in this Journal (Vol. II. pp. 71–84); he there mentions (p. 75) a question (which I have passed over) concerning the legitimacy of the mode adopted in expanding the partial fractions.

Another point of symmetry may be deduced from the above identities. For even quantics, comparing only the coefficients of the highest negative power of x on the two sides of the identity, we find

$$(-1)^n a^n A_0 - A_q = 0;$$

but we have seen that $A_q = (-1)^{n-k}A'_0$; therefore

$$A_0 = (-1)^k a^{-n}A'_0;$$

i. e. the invariantive part of the numerator has a symmetry of its own, the coefficient of a^λ being $(-1)^k$ times the coefficient of $a^{p-n-\lambda}$; of course a corresponding symmetry holds for A_q. For odd quantics we find in like manner that the coefficient of any term a^λ in A_0 is $(-1)^{k-1}$ times the coefficient of $a^{p-n-1-\lambda}$ in the same.

Sylvester's Second Method.

This method, like Professor Cayley's, is based upon the fact that we know *à priori* the denominator and the degrees of the numerator of the generating function. It was devised by Professor Sylvester shortly after Professor Cayley had invented the one just explained, and before the latter had made it known.

The process is extremely simple. If the generating function for a quantic of the order i were expanded, the coefficient of any term $a^j x^g$ in the expansion would be $\Delta(i, j : g)$. The coefficients of the expansion can therefore be obtained by formula (5), p. 129. Now the expansion multiplied by the known denominator of the generating function gives the numerator; and since the latter is confined to known limits of degree in a and in x (while the denominator contains no negative powers of a or x), we have to use only those terms of the expansion which fall within these limits and multiply their sum by the known denominator; that portion of the product which falls within the limits of degree is the required numerator.

The work, then, can be stated as follows. Let the order of the quantic be i, the denominator of the required generating function D, and the degrees in a and x of the numerator p and q. Develop the fractions of the form

$$\frac{(1-z^{j+1})(1-z^{j+2})\ldots\ldots(1-z^{j+i})}{(1-z^2)(1-z^3)\ldots\ldots(1-z^i)}$$

obtained by giving to j all values from $j=0$ to $j=p$. The coefficient of z^w in the development of the above fraction is $\Delta(w : i, j) = \Delta(i, j : ij - 2w)$ and is therefore the coefficient of $a^j x^{ij-2w}$ in the expansion of the generating function. If, then, ij is even, the coefficients of $z^{\frac{ij}{2}}$ and of the $\frac{1}{2}q$ (or $\frac{1}{2}(q-1)$) next preceding terms in the development of the above fraction are the coefficients of $a^j x^0, a^j x^2, \ldots\ldots, a^j x^q$ (or $a^j x^{q-1}$) in the expansion of the generating function; if ij is odd, the coefficients of $z^{\frac{ij-1}{2}}$ and of the $\frac{1}{2}(q-1)$ (or $\frac{1}{2}(q-2)$) next preceding terms are the coefficients of $a^j x^1, a^j x^3, \ldots\ldots, a^j x^q$ (or $a^j x^{q-1}$) in the expansion of the generating function. Multiplying the aggregate of the terms thus obtained by D, restricting the product to terms whose degrees do not surpass p, q we have the required numerator. Of course, we do not actually use values of j higher than p' $(=\frac{1}{2}p$ or $\frac{1}{2}(p+1))$, the remaining half of the numerator being obtained by symmetry.

As none of the larger calculations have been performed by more than one of the methods described, I cannot say positively whether this or Professor Cayley's method is practically the more desirable, but I believe that this method is to be preferred. It has the advantage of greater uniformity and less liability to accidental errors, and there is, I think, no very great difference (on which side I do not know) in the amount of numerical work required.

It may be well to mention at this place the great advantage of employing paper ruled into squares, in performing the calculations required by any of these

methods. For instance, in the *developments* required by the method last described, the only operations that occur are those of multiplication and division by factors of the form $1-z^\lambda$. The coefficients of the successive powers of z are written in the successive squares of a row: to multiply by $1-z^\lambda$, the same series of coefficients shoved λ places forward is subtracted from the given series; to divide by $1-z^\lambda$, the first λ terms of the given series are written down as the first λ terms of the quotient, and from this point on, the series of quotient-coefficients is continued by being shoved λ places forward and added to the given series. I find it convenient not to write down the figures to be subtracted or added; multiplication by $1-z^\lambda$ being performed by subtracting from each coefficient of the multiplicand the coefficient λ spaces back of it, and division by adding to each coefficient of the dividend the coefficient λ spaces back of it in the quotient: the regularity of the squares enables one to do this without danger of error. The advantage of this use of paper ruled into squares is even greater when powers of *two* letters (say a and x) are involved. The coefficients are then arranged in horizontal and vertical rows of squares according to the powers of a and x (as in the tables, this Journal, Vol. II. pp. 226–246); and, e. g., multiplication by $1-ax^3$ is performed by subtracting from each coefficient of the multiplicand the coefficient one space above and three spaces to the left of it. It is needless to go into further details on this point; a very great amount of time is saved by this practically almost indispensable contrivance, which any one who had occasion for it would know how to use to the best advantage.

It may be worth while to say a few words concerning the arrangement of the work (as actually performed) in Sylvester's second method. We first develop the fraction

$$\frac{(1-z^{p'+1})(1-z^{p'+2})\ldots.(1-z^{p'+i})}{(1-z^2)(1-z^4)\ldots.(1-z^i)}$$

(where p' is the highest value of j required) as far as it is needed; this result divided by $1-z^{p'+i}$ and multiplied by $1-z^{p'}$ gives the series corresponding to $j=p'-1$; this divided by $1-z^{p'+i-1}$ and multiplied by $1-z^{p'-1}$ gives the series corresponding to $j=p'-2$; we continue in this way, obtaining the series for each value of j from the next higher, till we reach $j=0$, for which the value of the fraction is $1-z$: and it is easy to see that (if each series is carried a few terms beyond the point actually needed) we have thus a perfect test of this portion of the work. In other parts of the work, obvious and easy partial tests can be made; and a very good test of the correctness of the result obtained for the numerator of the generating function is afforded by the symmetry of A_0 (see p. 146). For in the calculation of A_q all or nearly all the

coefficients in $A_0, A_1, A_2, \ldots. A_{q-1}$ are involved; now it is by means of A_q that (from the rule of symmetry) we complete A_0, and if this completion makes A_0 symmetrical, A_q and A_0 are perfectly tested, and consequently the intervening A's pretty thoroughly. Of course, if instead of calculating half the numerator we calculated the whole, its symmetry would be a perfect test, and no other test of the result would be needed; but in the longer cases this would multiply the labor many times. There are many short and satisfactory ways of checking the work in its progress, which would suggest themselves to any calculator, and which it would be tedious to detail. Another test of the *result*, which with the one above given puts its correctness beyond the region of practical doubt, will be mentioned in connection with the generating functions for differentiants.

Generating Functions for Differentiants.

If we put $x=1$ in the expansion of the generating function, the coefficient of a^j will be the number of linearly independent covariants of the degree j and of all orders; or, say, the number of linearly independent *differentiants* of the degree j (and of all weights). Hence by putting $x=1$ in the result given by any of the methods above described, we obtain what Professor Sylvester calls the *generating function for differentiants.*

But this generating function can also be obtained more directly. The number of linearly independent differentiants of the degree j and of all weights is

$$\Delta(W:i,j)+\Delta(W-1:i,j)+\Delta(W-2:i,j)+\ldots+\Delta(2:i,j)+\Delta(1:i,j)+\Delta(0:i,j),$$

where W is the highest weight which a differentiant of degree j can have, viz. $\frac{ij}{2}$ or $\frac{ij-1}{2}$; and it is easily seen that $\Delta(0:i,j)$ is to be regarded as 1. Writing this sum more explicitly

$$(W:i,j)-(W-1:i,j)+(W-1:i,j)-(W-2:i,j)+\ldots+(2:i,j)-(1:i,j)+(1:i,j)-(0:i,j)+1$$

we see that its value is $(W:i,j)$. Now $(W:i,j)$ is the coefficient of $a^j x^{ij-2W}$ in the development of

$$\frac{1}{(1-ax^i)(1-ax^{i-2})\ldots.(1-ax^{-i+2})(1-ax^{-i})}.$$

If i is even, $ij-2W$ is 0 for all values of j; if i is odd, $ij-2W$ is 0 or 1 according as j is even or odd. Accordingly, if i is even, the multiplier of x^0, and if i is odd, the sum of the multipliers of x^0 and x^1,* in the development of the above fraction, is the generating function for differentiants.

* When i is odd, even powers of x are multiplied only by even powers of a and odd powers only by odd powers, in the development; so that in this addition each power of a is taken only once, as it should be.

This generating function is calculated by means of the decomposition of the above fraction into partial fractions with respect to x, as in "Sylvester's First Method."

Suppose $i = 2n$. The fraction corresponding to $1 - ax^{2\lambda}$ (λ positive) is

$$\Sigma \frac{A}{1 - \rho a^{\frac{1}{\lambda}} x^2} = \Sigma \frac{A}{1 - \alpha x^2}, \text{ say};$$

where

$$\lambda A = (-)^{n-\lambda} \frac{a^{\frac{(n-\lambda)(n-\lambda+1)}{2}}}{(1-a)^2 (1-a^2)^2 (1-a^3)^2 \ldots (1-a^{n-\lambda})^2 (1-a^{n-\lambda+1}) \ldots (1-a^{n+\lambda})}.$$

In the development of this fraction, the multiplier of x^0 is simply ΣA; so that the generating function for differentiants will be obtained by adding together ΣA and the analogous expressions for all the factors of the denominator in which the exponent of x is positive. It is plain, from what has been given at p. 139, that these expressions will have for a common denominator

$$(1-a)(1-a^2)^2(1-a^3)(1-a^4) \ldots (1-a^{2n-1}).$$

The degree of the numerator is (see p. 140) less by $2n+1$ than that of the denominator.

Suppose $i = 2n+1$. The fraction corresponding to $1 - ax^{\lambda}$ (λ positive) is

$$\Sigma \frac{A}{1 - \rho a^{\frac{1}{\lambda}} x} = \Sigma \frac{A}{1 - \alpha x}, \text{ say};$$

where

$$\lambda A = (-)^{\frac{2n+1-\lambda}{2}} \frac{a^{\frac{2n+1-\lambda}{2} \cdot \frac{2n+3-\lambda}{2}}}{(1-a^2)^2 (1-a^4)^2 \ldots (1-a^{2n+1-\lambda})^2 (1-a^{2n+3-\lambda}) \ldots (1-a^{2n+1+\lambda})}.$$

In the development of this fraction, the sum of the multipliers of x^0 and x^1 is $\Sigma A (1+\alpha)$; and the generating function for differentiants is obtained by adding together $\Sigma A(1+\alpha)$ and the similar expressions corresponding to the several factors of the denominator in which the exponent of x is positive. These expressions will have for a common denominator (see p. 140)

$$(1-a)(1-a^2)(1-a^4)(1-a^6) \ldots (1-a^{4n}).$$

The degree of the numerator is (see p. 141) less by $2n+2$ than that of the denominator.

Thus the calculation of the generating function for differentiants is much shorter than that of the generating function for covariants; and the value of the former, thus independently obtained, affords a perfect test of the correctness of the latter: putting $x = 1$ in the generating function for covariants, the resulting fraction must be identical in value with the generating function for differentiants,

with which it can be compared with very little labor by reduction to a common denominator.—But for practical certainty it is not necessary to make an independent calculation of the generating function for differentiants at all; our knowledge of its *denominator* suffices to give a very searching test. The denominator of the generating function for covariants (in all but the lowest orders of quantics) becomes, on putting $x = 1$, the product of the denominator of the generating function for differentiants by a power of $1 - a$; the numerator of the generating function for covariants must therefore, on putting $x = 1$, become divisible by this power of $1 - a$; the performance of this division therefore at once very thoroughly tests the generating function for covariants and gives us the generating function for differentiants by a self-checking process, which renders the independent calculation of the latter unnecessary.

It seems needless to set forth two other methods which might be used for calculating the generating function for differentiants, having the same relation to the above as Cayley's method and Sylvester's second method of calculating the generating function for covariants have to Sylvester's first method.

Systems of Quantics.

The methods of obtaining the generating functions and groundforms for systems of quantics are extensions of the methods used for single quantics; it seems unnecessary to set them out at length, a very brief account being now sufficient to make them intelligible.

We must first define an extension of the notation $(w : i, j)$. The number of ways in which w can be composed by the addition of j numbers taken from the set $0, 1, 2, \ldots . i$, together with j' numbers from the set $0, 1, 2, \ldots . i'$, j'' from the set $0, 1, 2, \ldots . i''$, etc. is denoted by the symbol

$$(w : i, j; i', j'; i'', j''; \ldots .)$$

E. g., $(5 : 3, 4; 4, 2) = 21$, since 5 can be composed in the following 21 ways, using four of the numbers $0, 1, 2, 3$ and two of the numbers $0, 1, 2, 3, 4$:—

3200,00	3110,00	2210,00	2111,00	3100,10	2200,10
2110,10	1111,10	2100,20	2100,11	1110,20	1110,11
2000,30	2000,21	1100,30	1100,21	1000,40	1000,31
1000,22	0000,41	0000,32 *			

* It is obvious that $(w : i, j; i', j'; i'', j''; \ldots .) = \Sigma\Sigma\Sigma \ldots . (v : i, j)\,(v' : i', j')\,(v'' : i'', j'') \ldots .$; the summations referring to the v's, which are to take all positive integer values (including 0) subject to the condition $v + v' + v'' + \ldots . = w$.

The fundamental theorem on which the investigation of the generating functions and groundforms of a system of quantics rests is an extension of the fundamental theorem for a single quantic, and was demonstrated by Professor Sylvester along with the latter. It is as follows:—

The number of linearly independent differentiants of weight w and degrees $j, j', j'' \dots$ in the coefficients of a system of quantics of the orders $i, i', i'' \dots$ respectively is $(w : i, j; i', j'; i'', j''; \dots) - (w-1 : i, j; i', j'; i'', j''; \dots)$.

It follows, in the same manner as the corresponding theorem for single quantics, that the number of linearly independent covariants of order g and degrees $j, j', j'' \dots$ is the coefficient of $a^j b^{j'} c^{j''} \dots x^g$ in the development of

$$\frac{1-x^{-2}}{(1-ax^i)(1-ax^{i-2})\dots(1-ax^{-i})(1-bx^{i'})(1-bx^{i'-2})\dots(1-bx^{-i'})(1-cx^{i''})(1-cx^{i''-2})\dots(1-cx^{-i''})\dots}$$

in ascending powers of $a, b, c \dots$

If this fraction be decomposed into partial fractions with respect to x as in "Sylvester's First Method," we find, as in the case of a single quantic, that the partial fractions corresponding to those factors of the denominator in which the exponent of x is positive, are the only ones whose development in ascending powers of $a, b, c \dots$ gives non-negative powers of x; so that the generating function for the covariants of a system of quantics is obtained by adding together *these* partial fractions only. The process is thus an obvious extension of "Sylvester's First Method," which need not be further explained.

There is a feature of the calculation, however, that does not appear in the case of a single quantic. This will be sufficiently shown by an example; let us take the system of a quadric and a cubic. The crude form of the generating function is

$$\frac{1-x^{-2}}{(1-bx^2)(1-b)(1-bx^{-2})(1-cx^3)(1-cx)(1-cx^{-1})(1-cx^{-3})}.$$

The sum of the partial fractions corresponding to $1-bx^2$ has for its denominator

$$(1-b)(1-bc^2)(1-b^3c^2)(b-c^2)(b^3-c^2);$$

the like sums corresponding to $1-cx^3$ and $1-cx$ have for their denominators respectively

$$(1-b)(1-c^2)(1-c^4)(1-b^3c^2)(b^3-c^2),$$

and

$$(1-b)(1-c^2)(1-c^4)(1-bc^2)(b-c^2).$$

The sum of all these fractions, which is the generating function sought, would therefore appear to contain in its denominator the factors $b-c^2$ and b^3-c^2; and it is plain that the presence of these factors in the denominator would render the expansion of the fraction in simultaneously ascending powers of b and c

impossible. But as a matter of fact, the *numerator* obtained by adding the partial fractions also contains $b - c^2$ and $b^3 - c^2$ as factors; and on cancellation of them, the generating function has essentially the same character as that of a single quantic. The feature above exemplified appears in all cases of systems of quantics.

All the generating functions for systems of quantics that have as yet been obtained were calculated by the method above indicated. It is easy to see the course that would have to be pursued in basing upon it extensions of Cayley's method and of Sylvester's second method.

When the reduced form of the generating function has been obtained, it is converted into the representative form in a mode analogous to that employed for a single quantic.

The table of groundforms is deduced from the representative generating function by precisely the same process as for a single quantic. The validity of the process is dependent on the same postulate, and, granting that postulate, proved by the same reasoning as in the case of a single quantic. In fact, if we merely change *degorder* $(j \, . \, g)$ into *degorder* $(j \, , j' \, , j'' \, . \, . \, . \, . \, . \, g)$ and $a^r x^s$ into $a^r b^t c^u \, . \, . \, . \, . \, x^s$ throughout, every word of the argument under the head "Mode of obtaining the Table of Groundforms" applies to a system of any number of quantics.

Finally, it may be mentioned that the generating function for differentiants may be obtained from that for covariants by putting $x = 1$ in the latter. It may also be obtained independently (and in general in lower terms) by a process precisely analogous to that explained under the head "Generating Functions for Differentiants."